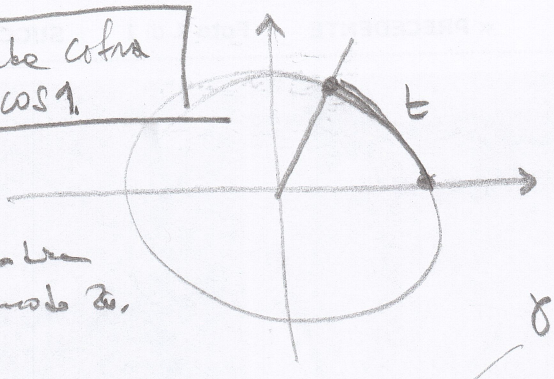
Trigonometria – numeri complessi – Equazioni differenziali

$\boxed{\text{Articolo: il coseno}}$

$[\cos 1 = 0.5403\ldots]$

$\boxed{\text{Sembra cotra di } \cos 1.}$



(1) Def. coseno?

$t \mapsto \cos t$ funzione periodica di periodo 2π.

$t \in \mathbb{R}$?

(gradi ?!).

t = angolo in radianti

(2) definizione tramite triangoli dettagli $\cos t$ $\quad 0<t<\frac{\pi}{2}$

$\cos t = \frac{a}{c}$

conoscenza di π

e per gli altri t.? misurati in?

(ii) Cerchio trigonometrico

$0 \le t < 2\pi :=$ "lunghezza" di S^1 orientata (verso antiorario)

$\forall t$ $\exists!$ arco $\gamma(t)$ di "lunghezza" t dal punto $(1,0)$ su S^1

lunghezza $t \longrightarrow \exists!\ (x,y) \in S^1$: gli estremi di

$\gamma(t)$ sono $(1,0)$ e (x,y)

Def. $\cos t := x$, $\sin t := y$.

prolungate periodicamente $\forall x \in \mathbb{R}$

Cos'è una funzione? $\quad x \in A \subseteq \mathbb{R} \mapsto f(x) \in \mathbb{R}$.

x^2, x^{27}, $\sqrt{x}$, e^x, $\log x$, $\cos x$, $\cot x$

$\frac{1+x}{1-x}$ definizione algebrica

Algoritmo : <2 —

$\sqrt[3]{3} = 1.4422.$ $\qquad 1^3 = 1 < 3 < 8 = 2^3$

$\left(\frac{3}{2}\right)^3 = \frac{27}{8} > 3 \qquad \left(\frac{3}{2}\right) > \sqrt[3]{3}$

$\left(\frac{5}{4}\right)^3 = \frac{125}{64} < 3$, etc. — —

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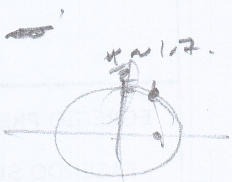
come $\underline{\cos 1} = ?$ (non $\cos 1°$) , $\underline{\cos 1}$. —



$0 < \cos x < 1$

$0,\_$ ↑ ?

DEF.

$$\cos x := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

$4! =$

$1 - \frac{1}{2} = \frac{1}{2}$

$$\left| \cos x - 1 + \frac{x^2}{2} \right| < \sum_{k=2}^{\infty} \frac{|x|^{2k}}{(2k)!}$$

$$\left| \cos 1 - \frac{1}{2} \right| < \sum_{2}^{\infty} \frac{1}{(2k)!} = \frac{1}{24} + \frac{1}{24 \cdot 5} + \frac{1}{24} \cdot \frac{1}{5 \cdot 6} + \frac{1}{24} \frac{1}{5 \cdot 6 \cdot 7}$$

$$= \frac{1}{24} + \frac{1}{120} + \frac{1}{24} \sum_{n=5}^{\infty} \frac{1}{n(n+1)} < \frac{1}{24}\left(1 + \frac{1}{5} + \frac{1}{5}\right)$$

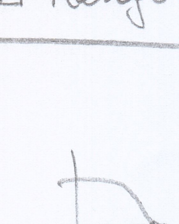
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \qquad \sum_{n=5}^{N} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} - \dots \to \frac{1}{5}$$

$$= \frac{1}{24}\left(1 + \frac{2}{5}\right) = \frac{1}{24} \cdot \frac{7}{5} = \frac{7}{120} < \frac{1}{10}$$

serie telescopica
(serie di Mengoli)

$\cos 1 = 0.5\dots$

Def. $\sin x = \sum_{k=0}^{\infty} (-1)^{2k+1} \frac{x^{2k+1}}{(2k+1)!}$



Def. $\frac{\pi}{2} :=$ primo 0 reale di $\cos x$.

$(\cos 0 = 1 ,\ \cos 2 < 0)$

- $\cos(x+y) = ?$

FORMULA di EULERO $\boxed{e^{i\pi}+1=0.}$

Numeri complessi

$e^{it} = \cos t + i \sin t \qquad \cos t = \mathrm{Re}\, e^{it}$

$\mathrm{Re}\left(e^{i(x+y)}\right) = \mathrm{Re}\left(e^{ix} e^{iy}\right) = \mathrm{Re}\left[(\cos x + i\sin x)(\cos y + i \sin y)\right]$

$\cos x \cos y - \sin x \sin y.$

formule di addizione per gli sinusoidi

$\mathbb{C} = (\mathbb{R}^2, *, +) \qquad (x_1, y_1) * (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$

[Ricorda prodotto in $\mathbb{Z}$: $[(x_1, y_1)] \cdot [(x_2, y_2)] = (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2)$]

- $(0,1) = i \qquad (0,1)*(0,1) = (-1, 0) = (-1,0),$

$(\mathbb{R}, *) \overset{(x,0)}{\cong} (\{(x,0) \mid x \in \mathbb{R}\}, *)$

Una strada per la formula di Euler

Funzione Esponenziale

$$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$= 1 + z + \frac{z^2}{2!} + \cdots$$

$\exp(0) = 1.$

$\forall z;\ \exists n:\ \frac{|z|}{n} < \frac{1}{2},\quad n > 2|z|$

$$\forall k \geq 0,\quad \frac{|z|^{n+k}}{(n+k)!} = \overbrace{\frac{|z| \cdots |z|}{(n+k) \cdots (n+1)}}^{k} \frac{|z|^n}{n!} \leq \frac{|z|^n}{n!}\left(\frac{|z|}{n}\right)^k < \underbrace{\frac{|z|^n}{n!}}_{B} \frac{1}{2^k}$$

$$\sum_0^{\infty}{}' = \sum_0^{n} + \sum_{n+1}^{\infty} = \underbrace{\sum_0^{n} \frac{|z|^k}{k!}}_{A} + \sum_{k=n+1}^{\infty} \frac{|z|^k}{(k)!} =$$

$$= A + \sum_{k=1}^{\infty} \frac{|z|^{n+k}}{(n+k)!} < A + B \sum_1^{\infty} \frac{1}{2^k} = A + B$$

Teorema di addizione.

$$\exp(z+w) = \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z^j w^{k-j}$$

$$= \sum_{k \geq 0} \sum_{0 \leq j \leq k} \frac{1}{j!(k-j)!} = \sum_{k} \sum_{0 \leq j \leq k} \frac{z^j}{j!} \frac{w^{(k-j)}}{(k-j)!}$$

$$= \sum_{j \geq 0} \frac{z^j}{j!} \sum_{k \geq j} \frac{w^{k-j}}{(k-j)!} = \sum_{j \geq 0} \frac{z^j}{j!} \left(\sum_{h \geq 0} \frac{w^h}{h!} \right)$$

$$= \exp(z) \exp(w).$$

Teorema $\boxed{\exp(z+w) = \exp(z) \cdot \exp(w).}$

- $\exp(it) = \cos t + i \sin t$ (formule)
- Formule di addizione per seno e coseno:

$$\exp(i(s+t)) = \exp(is)\exp(it) = (\cos s + i\sin s)(\cos t + i \sin t)$$

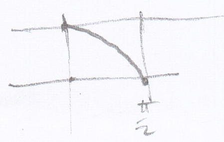
$$\cos(s+t) + i\sin(s+t) = (\cos s \cos t - \sin t \sin s) + i(\cos s \sin t + \sin s \cos t)$$

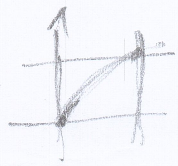
- $\overline{\exp(it)} = \exp(-it)$

$$\exp(it)\,\overline{\exp(it)} = \exp(0) = 1 \Rightarrow$$

$$|\exp(it)|^2$$

ovvero $\boxed{\cos^2 t + \sin^2 t = 1}$





$\sin t = t - \frac{t^3}{2} + \dots$

$\sin t > 0$ per t piccoli

- $\exp(i\frac{\pi}{2}) = \sin\frac{\pi}{2} \Rightarrow \sin\frac{\pi}{2} = 1.$

etc. $\cos\pi = \cos(\frac{\pi}{2}+\frac{\pi}{2}) = -\sin^2\frac{\pi}{2} = -1.$

$x \in \mathbb{R}$

- $\exp(x) = e^x$ se $e := \exp(1)$

Derivate

~~$\frac{\sin t}{t} \to 1 \quad \frac{\cos t - 1}{t^2} \to -\frac{1}{2}$~~

- $\frac{\sin(t+h) - \sin t}{h} = \frac{\sin t \cos h + \cos t \sin h - \sin t}{h}$

$$= \left(\frac{1-\cos h}{h}\right)\sin t + \cos t \frac{\sin h}{h} \to \cos t$$

- $\frac{\cos(t+h) - \cos t}{h} = \frac{\cos t \cos h - \sin t \sin h - \cos t}{h} \to -\sin t.$